On n-dependent groups and fields

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Abstract

First, an example of a 2-dependent group without a minimal subgroup of bounded index is given. Second, all infinite n-dependent fields are shown to be Artin-Schreier closed. Furthermore, the theory of any non separably closed PAC field has the IP_n property for all natural numbers n and certain properties of dependent (NIP) valued fields extend to the n-dependent context.

1 Introduction

Macintyre [18] and Cherlin-Shelah [6] have shown independently that any superstable field is algebraically closed. However, less is known in the case of supersimple fields. Hrushovski proved that any infinite perfect bounded pseudo-algebraically closed (PAC) field is supersimple [14] and conversely supersimple fields are perfect and bounded (Pillay and Poizat [19]), and it is conjectured that they are PAC. More is known about Artin-Schreier extensions of certain fields. Using a suitable chain condition for uniformly definable subgroups, Kaplan, Scanlon and Wagner showed in [16] that infinite NIP fields of positive characteristic are Artin-Schreier closed and simple fields have only finitely many Artin-Schreier extensions. The latter result was generalized to fields of positive characteristic defined in a theory without the tree property of the second kind (NTP_2 fields) by Chernikov, Kaplan and Simon [8].

We study groups and fields without the n-independence property. Theories without the n-independence property, briefly n-dependent or NIP_n theories, were introduced by Shelah in [21]. They are a natural generalization of NIP theories, and in fact both notions coincide when n equals to 1. For background on NIP theories the reader may consult [25]. It is easy to see that any theory with the $(n+1)$-independence property has the n-independence property. On the other hand, as for any natural number n the random $(n+1)$-hypergraph is $n+1$-dependent but has the n-independence property [9, Example 2.2.2], the classes of n-dependent theories form a proper hierarchy. Additionally, since all random hypergraphs are simple, the previous example shows that there are theories which are simple and n-dependent but which are not NIP. Hence one might ask if there are any non combinatorial examples of n-dependent theories which have the independence property? And furthermore, which results of NIP theories can be generalized to n-dependent theories or more specifically which results of (super)stable theories remains

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true for (super)simple n-dependent theories? Beyarslan [2] constructed the random n-hypergraph in any pseudo-finite field or, more generally, in any e-free perfect PAC field (PAC fields whose absolute Galois group is the profinite completion of the free group on e generators). Thus, those fields lie outside of the hierarchy of n-dependent fields.

In this paper, we first give an example of a group with a simple 2-dependent theory which has the independence property. Additionally, in this group the A-connected component depends on the parameter set A. This establishes on the one hand a non combinatorial example of a proper 2-dependent theory and on the other hand shows that the existence of an absolute connected component in any NIP group cannot be generalized to 2-dependent groups. Secondly, we find a Baldwin-Saxl condition for n-dependent groups (Section 4). Using this and connectivity of a certain vector group established in Section 5 we deduce that n-dependent fields are Artin-Schreier closed (Section 6). Furthermore, we show in Section 7 that the theory of any non separably closed PAC field has in fact the IP_n property for all natural numbers n which was established by Duret for the case n equals to 1 [10]. In Section 8 we extend certain consequences found in [16] for dependent valued fields with perfect residue field as well as in [15] by Jahnke and Koenigsmann for NIP henselian valued field to the n-dependent context.

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2 Preliminaries

In this section we introduce n-dependent theories and state some general facts. The following definition can be found in [22, Definition 2.4].

Definition 2.1. Let T be a theory. We say that a formula $\psi(\bar{y}_0, \dots, \bar{y}_{n-1}; \bar{x})$ in T has the *n-independence property* (IP_n) if there exists some parameters $(\bar{a}_i^j : i \in \omega, j \in n)$ and $(\bar{b}_I : I \subset \omega^n)$ in some model $\mathcal{M}$ of T such that $\mathcal{M} \models \psi(\bar{a}_{i_0}^0, \dots, \bar{a}_{i_{n-1}}^{n-1}, \bar{b}_I)$ if and only if $(i_0, \dots, i_{n-1}) \in I$.

A theory is said to have IP_n if one of its formulas has IP_n. Otherwise we called it *n-dependent.* A structure is said to have IP_n or to be n-dependent if its theory does.

Both facts below are useful in order to proof that a theory is n-dependent as it reduces the complexity of formulas one has to consider to have IP_n. The first one is stated as Remark 2.5 [22] and afterwards proved in detail as Theorem 6.4 [9].

Fact 2.2. *A theory T is n-dependent if and only if every formula $\phi(\bar{y}_0, ..., \bar{y}_{n-1}; x)$ with $|x| = 1$ is n-dependent.*

Fact 2.3. *[9, Corollary 3.15] Let $\phi(\bar{y}_0, ..., \bar{y}_{n-1}; \bar{x})$ and $\psi(\bar{y}_0, ..., \bar{y}_{n-1}; \bar{x})$ be n-dependent formulas. Then so are $\neg\phi$, $\phi \wedge \psi$ and $\phi \vee \psi$.*

Remark 2.4. Note that a formula with at most n free variables cannot witness the n-independence property. Thus, from the previous fact it is easy to deduce that the random

n-hypergraph is n-dependent. In fact, more generally any theory in which any formula of more than n free variables is a boolean combination of formulas with at most n free variables is n-dependent.

3 Example of a 2-dependent group without a minimal subgroup of bounded index

Let G be $\mathbb{F}_p^{(\omega)}$ where $\mathbb{F}_p$ is the finite field with p elements. We consider the structure $\mathcal{M}$ defined as $(G, \mathbb{F}_p, 0, +, \cdot)$ where 0 is the neutral element, $+$ is addition in G, and $\cdot$ is the bilinear form $(a_i)_i \cdot (b_i)_i = \sum_i a_i b_i$ from G to $\mathbb{F}_p$. This example in the case p equals 2 has been studied by Wagner in [26, Example 4.1.14]. He shows that it is simple and that the connected component G_A^0 for any parameter set A is equal to $\{g \in G : \bigcap_{a\in A} g \cdot a = 0\}$. Hence, it is getting smaller and smaller while enlarging A and whence the absolute connected component, which exists in any NIP group, does not for this example.

Lemma 3.1. *The theory of $\mathcal{M}$ eliminates quantifiers.*

Proof. Let $t_1(x;\bar{y})$ and $t_2(x;\bar{y})$ be two group terms in G and let ϵ be an element of $\mathbb{F}_p$. Observe that the atomic formula $t_1(x;\bar{y}) = t_2(x;\bar{y})$ (resp. $t_1(x;\bar{y}) \neq t_2(x;\bar{y})$) is equivalent to an atomic formula of the form $x = t(\bar{y})$ or $0 = t(\bar{y})$ (resp. $x \neq t(\bar{y})$ or $0 \neq t(\bar{y})$) for some group term $t(\bar{y})$. Note that $0 = t(\bar{y})$ as well as $0 \neq t(\bar{y})$ are both quantifier free formulas in the free variables $\bar{y}$. Furthermore, the atomic formulas $t_1(x;\bar{y}) \cdot t_2(x;\bar{y}) = \epsilon$ and $t_1(x;\bar{y}) \cdot t_2(x;\bar{y}) \neq \epsilon$ are equivalent to a boolean combination of atomic formulas of the form $x \cdot x = \epsilon_x$, $x \cdot t_i(\bar{y}) = \epsilon_i$ and $t_j(\bar{y}) \cdot t_k(\bar{y}) = \epsilon_{jk}$ (a quantifier free formula in the free variables $\bar{y}$) with $t_i(\bar{y})$ group terms and ϵ_x, ϵ_i, and ϵ_{jk} elements of $\mathbb{F}_p$. Thus, a quantifier free formula $\varphi(x,\bar{y})$ is equivalent to a finite disjunction of formulas of the form

$$\phi(x;\bar{y}) = \psi(\bar{y}) \wedge x \cdot x = \epsilon \wedge \bigwedge_{i\in I_0} x = t_i^0(\bar{y}) \wedge \bigwedge_{i\in I_1} x \neq t_i^1(\bar{y}) \wedge \bigwedge_{i\in I_2} x \cdot t_i^2(\bar{y}) = \epsilon_i$$

where $t_i^j(\bar{y})$ are group terms, ϵ, ϵ_i are elements of $\mathbb{F}_p$, and $\psi(\bar{y})$ is a quantifier free formula in the free variables $\bar{y}$. If I_0 is nonempty, the formula $\exists x \phi(x,\bar{y})$ is equivalent to

$$\psi(\bar{y}) \wedge \bigwedge_{j,l\in I_0} t_j^0(\bar{y}) = t_l^0(\bar{y}) \wedge t_i^0(\bar{y}) \cdot t_i^0(\bar{y}) = \epsilon \wedge \bigwedge_{j\in I_1} t_i^0(\bar{y}) \neq t_j^1(\bar{y}) \wedge \bigwedge_{j\in I_2} t_i^0(\bar{y}) \cdot t_j^2(\bar{y}) = \epsilon_j$$

for any $i \in I_0$. Now, we assume that I_0 is the empty set. If there exists an element x' such that $x' \cdot z_i = \epsilon_i$ for given $z_0, \ldots, z_m$ in G and $\epsilon_i \in \mathbb{F}_p$, one can always find an element x such that $x \cdot x = \epsilon$ and $x \neq v_j$ for given $v_0, \ldots, v_q$ in G which still satisfies $x \cdot z_i = \epsilon_i$ by modifying x' at a large enough coordinate. Hence, it is enough to find a quantifier free condition which is equivalent to $\exists x \bigwedge_{i\in I_2} x \cdot t_i^2(\bar{y}) = \epsilon_i$. For $i \in \mathbb{F}_p$, let

$$Y_i = \{j \in I_2 : \epsilon_j = i\}.$$

Then $\exists x \bigwedge_{i\in I_2} x \cdot t_i^2(\bar{y}) = \epsilon_i$ is equivalent to

$$\bigwedge_{i=0}^{p-1} \bigwedge_{j\in Y_i} t_j^2(\bar{y}) \notin \left\{ \sum_{k\in Y_0} \lambda_k^0 t_k^2(\bar{y}) + \cdots + \sum_{k\in Y_i\setminus j} \lambda_k^i t_k^2(\bar{y}) : \lambda_k^l \in \mathbb{F}_p, \sum_{l=1}^{i} \sum_{k\in Y_l}^{k\neq j} l \cdot_{\mathbb{F}_p} \lambda_k^l \neq i \right\}$$

which finishes the proof. ∎

Lemma 3.2. *The structure $\mathcal{M}$ is 2-dependent.*

Proof. We suppose, towards a contradiction, that $\mathcal{M}$ has IP_2. By Fact 2.2 we can find a formula $\phi(\bar{y}_0, \bar{y}_1; x)$ with $|x| = 1$ which witnesses the 2-independence property. By the proof of Lemma 3.1 and as being 2-dependent is preserved under boolean combinations (Fact 2.3), it suffices to prove that none of the following formulas can witness the 2-independence property in the variables $(\bar{y}_0, \bar{y}_1; x)$:

- quantifier free formulas of the form $\psi(\bar{y}_0, \bar{y}_1)$,
- the formula $x \cdot x = \epsilon$ with ϵ in $\mathbb{F}_p$,
- formulas of the form $x = t(\bar{y}_0, \bar{y}_1)$ for some group term $t(\bar{y}_0, \bar{y}_1)$,
- formulas of the form $x \cdot t(\bar{y}_0, \bar{y}_1) = \epsilon$ for some group term $t(\bar{y}_0, \bar{y}_1)$ and ϵ in $\mathbb{F}_p$.

As the atomic formula $\psi(\bar{y}_0, \bar{y}_1)$ does not depend on x and $x \cdot x = \epsilon$ does not depend on $\bar{y}_0$ nor $\bar{y}_1$ they cannot witness the 2-independence property in the variables $(\bar{y}_0, \bar{y}_1; x)$. Furthermore, as for given $\bar{a}$ and $\bar{b}$, the formula $x = t(\bar{a}, \bar{b})$ can be only satisfied by a single element, such a formula is as well 2-dependent. Thus the only candidate left is a formula of the form $x \cdot t(\bar{y}_0, \bar{y}_1) = \epsilon$ with $t(\bar{y}_0, \bar{y}_1)$ some group term in G and ϵ an element of $\mathbb{F}_p$. Thus, we suppose that the formula $x \cdot t(\bar{y}_0, \bar{y}_1) = \epsilon$ has IP_2 and choose some elements $\{\bar{a}_i : i \in \omega\}$, $\{\bar{b}_i : i \in \omega\}$ and $\{c_I : I \subset \omega^2\}$ which witnesses it. As $t(\bar{y}_0, \bar{y}_1)$ is just a sum of elements of the tuple $\bar{y}_0$ and $\bar{y}_1$ and G is commutative, we may write this formula as $x \cdot (t_a(\bar{y}_0) + t_b(\bar{y}_1)) = \epsilon$ in which the term $t_a(\bar{y}_0)$ (resp. $t_b(\bar{y}_1)$) is a sum of elements of the tuple $\bar{y}_0$ (resp. $\bar{y}_1$). Let

$$S_{ij} := \{x : x \cdot (t_a(\bar{a}_i) + t_b(\bar{b}_j)) = \epsilon\}$$

be the set of realizations of the formula $x \cdot (t_a(\bar{a}_i) + t_b(\bar{b}_j)) = \epsilon$. Note, that an element c belongs to S_{ij} if and only if we have that $e_{ij}(c)$ defined as

$$e_{ij}(c) = c \cdot \big(t_a(\bar{a}_i) + t_b(\bar{b}_j)\big)$$

is equal to ϵ. Let i, l, j, and k be arbitrary natural numbers. Then,

$$\begin{aligned} e_{ij}(c) &= c \cdot \big(t_a(\bar{a}_i) + t_b(\bar{b}_j)\big) \\ &= c \cdot \big((t_a(\bar{a}_i) + t_b(\bar{b}_k)) + (p-1)(t_a(\bar{a}_l) + t_b(\bar{b}_k)) + (t_a(\bar{a}_l) + t_b(\bar{b}_j))\big) \\ &= e_{ik}(c) + (p-1)e_{lk}(c) + e_{lj}(c). \end{aligned}$$

If the element c belongs to $S_{ik} \cap S_{lk} \cap S_{lj}$, the terms $e_{ik}(c)$, $e_{lk}(c)$, and $e_{lj}(c)$ are all equal to ϵ. By the equality above we get that $e_{ij}(c)$ is also equal to ϵ and so c also belongs to S_{ij}.

Let $I = \{(1,1), (1,2), (2,2)\}$. Then $c_I \in S_{22} \cap S_{12} \cap S_{11}$ but $c_I \notin S_{21}$ which contradicts the precious paragraph letting i and k be equal to 2 and l and j be equal to 1. Thus the formula $x \cdot t(\bar{y}_0, \bar{y}_1) = \epsilon$ is 2-dependent, hence all formulas in the theory of $\mathcal{M}$ are 2-dependent and whence $\mathcal{M}$ is 2-dependent. $\square$

4 Baldwin-Saxl condition for n-dependent theories

We shall now prove a suitable version of the Baldwin-Saxl condition [1] for n-dependent formulas.

Proposition 4.1. *Let G be a group and let $\psi(\bar{y}_0, \ldots, \bar{y}_{n-1}; x)$ be a n-dependent formula for which the set $\psi(\bar{b}_0, \ldots, \bar{b}_{n-1}; G)$ defines a subgroup of G for any parameters $\bar{b}_0, \ldots, \bar{b}_{n-1}$. Then there exists a natural number m_ψ such that for any d greater or equal to m_ψ and any array of parameters $(\bar{a}_{i,j} : i < n,\ j \leq d)$ there is $\nu \in d^n$ such that*

$$\bigcap_{\eta \in d^n} H_\eta = \bigcap_{\eta \in d^n, \eta \neq \nu} H_\eta$$

where H_η is defined as $\psi(\bar{a}_{0,i_0}, \ldots, \bar{a}_{n-1,i_{n-1}}; x)$ for $\eta = (i_0, \ldots, i_{n-1})$.

Proof. Suppose, towards a contradiction, that for an arbitrarily large natural number m one can find a finite array $(\bar{a}_{i,j} : i < n,\ j \leq m)$ of parameters such that $\bigcap_{\eta \in m^n} H_\eta$ is strictly contained in any of its proper subintersections. Hence, for every $\nu \in m^n$ there exists c_ν in $\bigcap_{\eta \neq \nu} H_\eta \setminus \bigcap_\eta H_\eta$.

Now, for any subset J of m^n, we let $c_J := \prod_{\eta \in J} c_\eta$. Note that $c_J \in H_\nu$ whenever $\nu \in m^n \setminus J$. On the other hand, if ν is an element of J, all factors of the product except of c_ν belong to H_ν, whence $c_J \notin H_\nu$. By compactness, one can find an infinite array of parameters $(\bar{a}_{i,j} : i < n,\ j \leq \omega)$ and elements $\{c_J : J \subset \omega^n\}$ such that c_J belongs to H_ν if and only if $\nu \notin J$. Hence, the formula $\neg\psi(\bar{y}_0, \ldots, \bar{y}_{n-1}; x)$ has IP_n and whence by Fact 2.3 the original formula $\psi(\bar{y}_0, \ldots, \bar{y}_{n-1}; x)$ has IP_n as well contradicting the assumption. $\square$

5 A special vector group

For this section, we fix an algebraically closed field $\mathbb{K}$ of characteristic $p > 0$ and we let $\wp(x)$ be the additive homomorphism $x \mapsto x^p - x$ on $\mathbb{K}$.

We analyze the following algebraic subgroups of $(\mathbb{K}, +)^n$:

Definition 5.1. For a singleton a in $\mathbb{K}$, we let G_a be equal to $(\mathbb{K}, +)$, and for a tuple $\bar{a} = (a_0, \ldots, a_{n-1}) \in \mathbb{K}^n$ with $n > 1$ we define:

$$G_{\bar{a}} = \{(x_0, \ldots, x_{n-1}) \in \mathbb{K}^n :\ a_0 \cdot \wp(x_0) = a_i \cdot \wp(x_i) \text{ for } 0 \leq i < n\}.$$

Recall that for an algebraic group G, we denote by G^0 the connected component of the unit element of G. Note that if G is definable over some parameter set A, its connected component G^0 coincides with the smallest A-definable subgroup of G of finite index.

Our aim is to show that $G_{\bar{a}}$ is connected for certain choices of $\bar{a}$, namely $G_{\bar{a}}$ coincides with $G^0_{\bar{a}}$.

Lemma 5.2. *Let k be an algebraically closed subfield of $\mathbb{K}$, let G be a k-definable connected algebraic subgroup of $(\mathbb{K}^n, +)$ and let f be a k-definable homomorphism from*

G to $(\mathbb{K},+)$ such that for every $\bar{g} \in G$ there are polynomials $P_{\bar{g}}(X_0,\ldots,X_{n-1})$ and $Q_{\bar{g}}(X_0,\ldots,X_{n-1})$ in $k[X_0,\ldots,X_{n-1}]$ such that

$$f(\bar{g}) = \frac{P_{\bar{g}}(\bar{g})}{Q_{\bar{g}}(\bar{g})}.$$

Then f is an additive polynomial in $k[X_0,\ldots,X_{n-1}]$. In fact, there exists natural numbers $m_0,\ldots,m_n$ such that f is of the form $\sum_{i=0}^{m_0} a_{i,0}X_0^{p^i} + \cdots + \sum_{i=0}^{m_n} a_{i,n}X_n^{p^i}$ with coefficients $a_{i,j}$ in k.

Proof. By compactness, one can find finitely many definable subsets D_i of G and polynomials $P_i(X_0,\ldots,X_{n-1})$ and $Q_i(X_0,\ldots,X_{n-1})$ in $k[X_0,\ldots,X_{n-1}]$ such that f is equal to $P_i(\bar{x})/Q_i(\bar{x})$ on D_i. Using [3, Lemma 3.8] we can extend f to a k-definable homomorphism $F : (\mathbb{K}^n,+) \to (\mathbb{K},+)$ which is also locally rational. Now, the functions

$$F_0(X) := F(X,0,\ldots,0),\ \ldots\ , F_{n-1}(X) := F(0,\ldots,0,X)$$

are k-definable homomorphisms of $(\mathbb{K},+)$ to itself. Additionally, they are rational on a finite definable decomposition of $\mathbb{K}$, so they are rational on a cofinite subset of $\mathbb{K}$. Hence every F_i is an additive polynomial in $k[X]$. Thus

$$F(X_0,\ldots,X_{n-1}) = F_0(X_0) + \cdots + F_{n-1}(X_{n-1})$$

is an additive polynomial in $k[X_0,\ldots,X_{n-1}]$ as it is a sum of additive polynomials. By [12, Proposition 1.1.5] it is of the desired form. □

Lemma 5.3. *Let $\bar{a} = (a_0,\ldots,a_n)$ be a tuple in $\mathbb{K}^\times$. Then $G_{\bar{a}}$ is connected if and only if the set $\left\{\frac{1}{a_0},\ldots,\frac{1}{a_n}\right\}$ is linearly $\mathbb{F}_p$-independent.*

Parts of the proof follows the one of [16, Lemma 2.8].

Proof. So suppose first that $\left\{\frac{1}{a_0},\ldots,\frac{1}{a_n}\right\}$ is linearly $\mathbb{F}_p$-dependent. Thus we can find elements $b_0,\ldots,b_{n-1}$ in $\mathbb{F}_p$ such that

$$b_0 \cdot \frac{1}{a_0} + \cdots + b_{n-1}\frac{1}{a_{n-1}} = \frac{1}{a_n}.$$

Now, let $\bar{a}'$ be the tuple $\bar{a}$ restricted to its first n coordinates and fix some element $(x_0,\ldots x_{n-1})$ in $G_{\bar{a}'}$. Let t be defined as $a_0(x_0^p - x_0)$. Hence, by the definition of $G_{\bar{a}'}$, we have that t is equal to $a_i(x_i^p - x_i)$ for any $i < n$. Furthermore, we have that $(x_0,\ldots,x_{n-1},x)$ belongs to $G_{\bar{a}}$ if and only if

$$\begin{aligned}
& t && = a_n(x^p - x)\\
\Leftrightarrow\ & 0 && = \frac{1}{a_n}t - (x^p - x)\\
\Leftrightarrow\ & 0 && = \frac{b_0}{a_0}t + \cdots + \frac{b_{n-1}}{a_{n-1}}t - (x^p - x)\\
\Leftrightarrow\ & 0 && = b_0 \cdot (x_0^p - x_0) + \cdots + b_{n-1} \cdot (x_{n-1}^p - x_{n-1}) - (x^p - x)\\
\Leftrightarrow\ & 0 && = (b_0 \cdot x_0 + \cdots + b_{n-1} \cdot x_{n-1} - x)^p - (b_0 \cdot x_0 + \cdots + b_{n-1}x_{n-1} - x).
\end{aligned}$$

In other words, $(x_0, \ldots, x_{n-1}, x)$ belongs to G_a if and only if $b_0 \cdot x_0 + \cdots + b_{n-1}x_{n-1} - x$ is an element of $\mathbb{F}_p$. With this formulation we consider the following subset of $G_{\bar{a}}$:

$$H = \{(x_0, \ldots x_n) \in G_{\bar{a}} : (x_0, \ldots x_{n-1}) \in G_{\bar{a}'} \text{ and } b_0 \cdot x_0 + \ldots b_{n-1}x_{n-1} - x_n = 0\}$$

This is in fact a definable subgroup of $G_{\bar{a}}$ of finite index. Hence $G_{\bar{a}}$ is not connected.

We prove the other implication by induction on the length of the tuple $\bar{a}$ which we denote by n. Let $n = 1$, then $G_{\bar{a}}$ is equal to $(\mathbb{K}, +)$ and thus connected since the additive group of an algebraically closed field is always connected.

Let $\bar{a} = (a_0, \ldots, a_n)$ be an $(n+1)$-tuple such that $\left\{\frac{1}{a_0}, \ldots, \frac{1}{a_n}\right\}$ is linearly $\mathbb{F}_p$-independent and suppose that the statement holds for tuples of length n. Define $\bar{a}'$ to be the restriction of $\bar{a}$ to the first n coordinates. Observe that the natural map $\pi : G_{\bar{a}} \to G_{\bar{a}'}$ is surjective since $\mathbb{K}$ is algebraically closed and that

$$[G_{\bar{a}'} : \pi(G_{\bar{a}}^0)] = [\pi(G_{\bar{a}}) : \pi(G_{\bar{a}}^0)] \leq [G_{\bar{a}} : G_{\bar{a}}^0] < \infty.$$

Hence the definable group $\pi(G_{\bar{a}}^0)$ has finite index in $G_{\bar{a}'}$. As $\left\{\frac{1}{a_0}, \ldots, \frac{1}{a_{n-1}}\right\}$ is also linearly $\mathbb{F}_p$-independent, the group $G_{\bar{a}'}$ is connected by assumption. Therefore $\pi(G_{\bar{a}}^0) = G_{\bar{a}'}$.

Now, suppose that $G_{\bar{a}}$ is not connected.

Claim. *For every $\bar{x} \in G_{\bar{a}'}$, there exists a unique $x_n \in \mathbb{K}$ such that $(\bar{x}, x_n) \in G_{\bar{a}}^0$.*

Proof of the Claim. Assume there exists $\bar{x} \in \mathbb{K}^n$ and two distinct elements x_n^0 and x_n^1 of $\mathbb{K}$ such that $(\bar{x}, x_n^0)$ and $(\bar{x}, x_n^1)$ are elements of $G_{\bar{a}}^0$. As $G_{\bar{a}}^0$ is a group, their difference $(\bar{0}, x_n^0 - x_n^1)$ belongs also to $G_{\bar{a}}^0$. Thus, by definition of $G_{\bar{a}}$, its last coordinate $x_n^0 - x_n^1$ lies in $\mathbb{F}_p$. So $(\bar{0}, \mathbb{F}_p)$ is a subgroup of $G_{\bar{a}}^0$. Take an arbitrary element $(\bar{x}, x_n)$ in $G_{\bar{a}}$. As $\pi(G_{\bar{a}}^0) = G_{\bar{a}'}$, there exists $x_n' \in \mathbb{K}$ with $(\bar{x}, x_n') \in G_{\bar{a}}^0$. Again, the difference of the last coordinate $x_n' - x_n$ lies in $\mathbb{F}_p$. So

$$(\bar{x}, x_n) = (\bar{x}, x_n') - (\bar{0}, x_n' - x_n) \in G_{\bar{a}}^0.$$

This leads to a contradiction, as $G_{\bar{a}}^0$ is assumed to be a proper subgroup of $G_{\bar{a}}$. □

Thus, we can fix a definable additive function $f : G_{\bar{a}'} \to \mathbb{K}$ that sends every tuple to this unique element. Note that $G_{\bar{a}}$ and hence also $G_{\bar{a}}^0$ are defined over $\bar{a}$. So the function f is defined over $\bar{a}$ as well. Now, let $\bar{x} = (x_0, \ldots, x_{n-1})$ be any tuple in $G_{\bar{a}'}$ and set $L := \mathbb{F}_p(a_0, \ldots, a_n)$. Then:

$$x_n := f(\bar{x}) \in \operatorname{dcl}(\bar{a}, \bar{x}).$$

In other words, x_n is definable over $L(x_0, \ldots, x_{n-1})$ which simply means that it belongs to the purely inseparable closure $\bigcup_{n\in\mathbb{N}} L(x_0, \ldots, x_{n-1})^{p^{-n}}$ of $L(x_0, \ldots, x_{n-1})$ by [5, Chapter 4, Corollary 1.4]. Since there exists an $l \in L(x_0)$ such that $x_n^p - x_n - a_n^{-1}l = 0$, the element x_n is separable over $L(x_0, \ldots, x_{n-1})$. So it belongs to $L(x_0, \ldots, x_{n-1})$ which implies that there exists some mutually prime polynomials $g, h \in L[X_0, \ldots, X_{n-1}]$ such that $x_n = h(x_0, \ldots, x_{n-1})/g(x_0, \ldots, x_{n-1})$. Thus, by Lemma 5.2 the definable function $f(X_0, \ldots, X_{n-1})$ we started with is an additive polynomial in n variables over L^{alg} and there exists $c_{j,i}$ in L^{alg} and natural numbers m_j such that

$$f(X_0, \ldots, X_{n-1}) = \sum_{i=0}^{m_0} c_{0,i} X_0^{p^i} + \cdots + \sum_{i=0}^{m_{n-1}} c_{n-1,i} X_{n-1}^{p^i}.$$

Using the identities $X_i^p - X_i = \frac{a_0}{a_i}(X_0^p - X_0)$ in $G_{\bar{a}}^0$, there are β_j in L^{alg} and $g(X_0) = \sum_{i=1}^{m_0} d_i X_0^{p^i}$ an additive polynomial in $L^{\mathrm{alg}}[X_0]$ with summands of powers of X_0 greater or equal to p such that

$$f(X_0, \ldots, X_{n-1}) = g(X_0) + \sum_{j=0}^{n-1} \beta_j \cdot X_j.$$

Since the image under f of the vectors $(0,1,0,\ldots,0), (0,0,1,0,\ldots,0), \ldots, (0,\ldots,0,1)$ has to be an element of $\mathbb{F}_p$, for $0 < i < n$ the β_i's have to be elements of $\mathbb{F}_p$. On the other hand, for any element $(x_0, \ldots, x_n)$ of $G_{\bar{a}}^0$ we have that $a_n(x_n^p - x_n) = a_0(x_0^p - x_0)$. Replacing x_n by $f(x_0, \ldots, x_{n-1})$ we obtain

$$\begin{aligned} 0 &= a_n \left[f(x_0, \ldots, x_{n-1})^p - f(x_0, \ldots, x_{n-1}) \right] - a_0(x_0^p - x_0) \\ &= a_n \left[g(x_0)^p - g(x_0) + (\beta_0^p x_0^p - \beta_0 x_0) + \sum_{j=1}^{n-1} \beta_j (x_j^p - x_j) \right] - a_0(x_0^p - x_0). \end{aligned}$$

Using again the identities $x_i^p - x_i = \frac{a_0}{a_i}(x_0^p - x_0)$ in $G_{\bar{a}}^0$ we obtain a polynomial in one variable

$$P(X) = a_n \left[g(X)^p - g(X) + (\beta_0^p X^p - \beta_0 X) + \sum_{j=1}^{n-1} \beta_j \frac{a_0}{a_j}(X^p - X) \right] - a_0(X^p - X)$$

which vanishes for all elements x_0 of $\mathbb{K}$ such that there exists $x_1, \ldots, x_{n-1}$ in $\mathbb{K}$ with $(x_0, \ldots, x_{n-1}) \in G_{\bar{a}'}$. In fact, this is true for all elements of $\mathbb{K}$. Hence, P is the zero polynomial. Notice that $g(X)$ appears in a pth-power. Since it contains only summands of power of X greater or equal to p, the polynomial $g(X)^p$ contains only summands of power of X strictly greater than p. As X only appears in powers less or equal to p in all other summands of P, the polynomial $g(X)$ has to be the zero polynomial itself. By the same argument as for the other β_j, the coefficient β_0 has to belong to $\mathbb{F}_p$ as well. Dividing by $a_0 a_n$ yields that

$$\sum_{j=0}^{n} \beta_j \frac{1}{a_j}(X^p - X)$$

with $\beta_n := -1$ is the zero polynomial. Thus

$$\sum_{j=0}^{n} \beta_j \frac{1}{a_j} = 0$$

As β_n is different from 0 and all β_i are elements of $\mathbb{F}_p$, this contradicts the assumption and the lemma is established. ☐

Using Lemma 5.3, a stronger version of [16, Lemma 2.8] together with [16, Corollary 2.6], we obtain the following corollary in the same way as Kaplan, Scanlon and Wagner obtain [16, Corollary 2.9].

Corollary 5.4. *Let k be a perfect subfield of $\mathbb{K}$ and $\bar{a} \in k^n$ be as in the previous lemma. Then $G_{\bar{a}}$ is isomorphic over k to $(\mathbb{K}, +)$. In particular, for any field $K \geq k$ with $K \leq \mathbb{K}$, the group $G_{\bar{a}}(K)$ is isomorphic to $(K, +)$.*

6 Artin-Schreier extensions

Definition 6.1. Let K be a field of characteristic $p > 0$ and $\wp(x)$ the additive homomorphism $x \mapsto x^p - x$. A field extension L/K is called an *Artin-Schreier extension* if $L = K(a)$ with $\wp(a) \in K$. We say that K is *Artin-Schreier closed* if it has no proper Artin-Schreier extension i. e. $\wp(K) = K$.

In the following remark, we produce elements from an algebraically independent array of size m^n which fit the condition of Lemma 5.3.

Remark 6.2. Let $\{\alpha_{i,j} : i \in n, j \in m\}$ be a set of algebraically independent elements in $\mathbb{K}$. Then the tuple $(a_{(i_0,\ldots,i_{n-1})} : (i_0, \ldots, i_{n-1}) \in m^n)$ with $a_{(i_0,\ldots,i_{n-1})} = \prod_{l=0}^{n-1} \alpha_{l,i_l}$ and ordered lexicographically satisfies the condition of Lemma 5.3.

Proof. Suppose that there exists a tuple of elements $(\beta_{(i_0,\ldots,i_{n-1})} : (i_0, \ldots, i_{n-1}) \in m^n)$ in $\mathbb{F}_p$ not all equal to zero such that

$$\sum_{(i_0,\ldots,i_{n-1})\in m^n} \beta_{(i_0,\ldots,i_{n-1})} \frac{1}{a_{(i_0,\ldots,i_{n-1})}} = 0$$

Then the $\alpha_{i,j}$ satisfy:

$$\sum_{(i_0,\ldots,i_{n-1})\in m^n} \beta_{(i_0,\ldots,i_{n-1})} \cdot \left(\prod_{\{(k,l)\neq(j,i_j):j\leq n-1\}} \alpha_{k,l} \right) = 0$$

which contradicts the algebraic independence of the $\alpha_{i,j}$. $\square$

We can now adapt the proof in [16] showing that an infinite NIP field is Artin-Schreier closed to obtain the same result for a n-dependent field.

Theorem 6.3. *Any infinite n-dependent field is Artin-Schreier closed.*

Proof. Let K be an infinite n-dependent field and we may assume that it is $\aleph_0$-saturated. We work in a big algebraically closed field $\mathbb{K}$ that contains all objects we will consider. Let $k = \bigcap_{l\in\omega} K^{p^l}$, which is a type-definable infinite perfect subfield of K. We consider the formula $\psi(x; y_0, \ldots, y_{n-1}) := \exists t\ (x = \prod_{i=0}^{n-1} y_i \cdot \wp(t))$ which for every tuple $(a_0, \ldots, a_{n-1})$ in k^n defines an additive subgroup of $(K, +)$. Let $m \in \omega$ be the natural number given by Proposition 4.1 for this formula. Now, we fix an array of size m^n of algebraically independent elements $\{\alpha_{i,j} : i \in n, j \in m\}$ in k and set $a_{(i_0,\ldots,i_{n-1})} = \prod_{l=0}^{n} \alpha_{l,i_l}$. By choice of m, there exists $(j_0, \ldots, j_{n-1}) \in m^n$ such that

$$\bigcap_{(i_0,\ldots,i_{n-1})\in m^n} a_{(i_0,\ldots,i_{n-1})} \cdot \wp(K) = \bigcap_{(i_0,\ldots,i_{n-1})\neq(j_0,\ldots,j_{n-1})} a_{(i_0,\ldots,i_{n-1})} \cdot \wp(K). \tag{6.1}$$

By reordering the elements, we may assume that $(j_0, \ldots, j_{n-1}) = (m, \ldots, m)$. Let $\bar{a}$ be the tuple $(a_{(i_0,\ldots,i_{n-1})} : (i_0, \ldots, i_{n-1}) \in m^n)$ ordered lexicographically and $\bar{a}'$ the restriction to $m^n - 1$ coordinates (one coordinate less).

We consider the groups $G_{\bar{a}}$ and respectively $G_{\bar{a}'}$ defined as in Definition 5.1. Using Remark 6.2 and Corollary 5.4 we obtain the following commuting diagram.

$$\begin{tikzcd} G_{\bar{a}} \arrow[r, "\pi"] \arrow[d, "\simeq"] & G_{\bar{a}'} \arrow[d, "\simeq"] \\ (\mathbb{K},+) \arrow[r, "\rho"] & (\mathbb{K},+) \end{tikzcd}$$

As the vertical isomorphisms are defined over k, this diagram can be restricted to K. Note that π and therefore also ρ stays onto for this restriction by equality (6.1) and that the size of $\ker(\rho)$ has to be p. Choose a nontrivial element c in the kernel of ρ and let ρ' be equal to $\rho(c \cdot x)$. Observe that ρ' is still a morphism from $(\mathbb{K},+)$ to $(\mathbb{K},+)$, its restriction to K is still onto and its kernel is equal to $\mathbb{F}_p$. Then [16, Remark 4.2] ensures that ρ' is of the form $a \cdot (x^p - x)^{p^n}$ for some a in K. Finally, let $l \in K$ be arbitrary. Since $\rho' \upharpoonright K$ is onto and X^{p^n} is an inseparable polynomial in characteristic p, there exists $h \in K$ with $l = h^p - h$. As $l \in K$ was arbitrary, we get that $\wp(K) = K$ and we can conclude. $\square$

The proof of [16, Corollary 4.4] adapts immediately and yields the following corollary.

Corollary 6.4. *If K is an infinite n-dependent field of characteristic $p > 0$ and L/K is a finite separable extension, then p does not divide $[L : K]$.*

7 Non separably closed PAC field

The goal of this section is to generalize a result of Duret [10], namely that the theory of a non separably closed PAC field has the IP property. To do so we need the following two facts.

Fact 7.1. *[10, Lemme 6.2] Let K be a field and k be a subfield of K which is PAC. Let p be a prime number which does not coincide with the characteristic of K such that k contains all pth roots of unity and there exists an element in k that does not have a pth root in K. Let $(a_i : i \in \omega)$ be a set of pairwise different elements of k and let I and J be finite disjoint subsets of ω, then K realizes*

$$\{\exists y(y^p = x + a_i) : i \in I\} \cup \{\neg\exists y(y^p = x + a_j) : j \in J\}.$$

Fact 7.2. *[10, Lemme 2.1] Every finite separable extension of a PAC field is PAC.*

Theorem 7.3. *Let K be a field and k be a subfield of K which is a non separably closed PAC field and relatively algebraically closed in K. Then, the theory of K has the n-independence property.*

Proof. If k is countable, we may work in an elementary extension of the tuple (K, k) for which it is uncountable. As k is non separably closed, there exists a proper Galois extension l of k. Let p be a prime number that divides the degree of l over k. Then there is a separable extension k' of k such that the Galois extension l over k' is of degree p. We may distinguish two cases:

(1) The characteristic of k is equal to p. As l is a cyclic Galois extension of degree p of k', a field of characteristic p, it is an Artin-Schreier extension of k'. We pick α such that $k' = k(\alpha)$ and let $K' = K(\alpha)$. As k' is separable over k, it is relatively algebraically closed in K' by [17, p.59]. Hence K' admits an Artin-Schreier extension and consequently its theory has IP_n by Theorem 6.3. As it is an algebraic extension of K, thus interpretable in K, the theory $\mathrm{Th}(K)$ has IP_n as well.

(2) The characteristic of k is different than p. Since l is a separable extension of k', we can find an element β of l such that l is equal to $k'(\beta)$. Let ω be a primitive p-root of unity and let $k'_\omega = k'(\omega)$ and $l_\omega = l(\omega)$. Note that l_ω is equal to $k'_\omega(\beta)$ and that the degree $[l_\omega : k'_\omega]$ is at most p and the degree $[k'_\omega : k']$ is strictly smaller than p. Additionally, we have:

$$[l_\omega : k'_\omega] \cdot [k'_\omega : k'] = [l_\omega : k'] = [l_\omega : l] \cdot [l : k'] = [l_\omega : l] \cdot p.$$

Thus $[l_\omega : k'_\omega]$ is divisible by p and hence equal to p. Furthermore, the conjugates of β over k'_ω are the same as over k'. Hence, as l is a Galois extension of k', they are contained in l and whence in l_ω. Thus, the field l_ω is a cyclic Galois extension of the field k'_ω and k'_ω contains the p-roots of unity. In other words, l_ω is a Kummer extension of k'_ω of degree p. So there exists an element δ in k'_ω that does not have a p root in it. Furthermore, as k'_ω is a finite separable extension of k, it is also PAC by Fact 7.2 and it is relatively algebraically closed in $K'_\omega = K'(\omega)$ by [17, p.59]. Thus, the element δ has no p-root in K'_ω as well. Let $\{a_{i,j} : j < n, i \in \omega\}$ be a set of algebraic independent elements of k'_ω which exists as it is an uncountable field. This ensures that $\prod_{l=0}^{n-1} a_{i_l,l} \neq \prod_{l=0}^{n-1} a_{j_l,l}$ for $(i_0, \dots, i_{n-1}) \neq (j_0, \dots, j_{n-1})$. Thus we may apply Fact 7.1 to K'_ω, k'_ω and the infinite set $\{\prod_{l=0}^{n-1} a_{i_l,l} : (i_0, \dots, i_{n-1}) \in \mathbb{N}^n\}$. We deduce that for the formula $\varphi(y; x_0, \dots, x_{n-1}) = \exists z(z^p = y + \prod_{i=0}^{n-1} x_i)$ and for any disjoint finite subsets I and J of $\mathbb{N}^n$ there exists an element in K'_ω that realizes

$$\{\varphi(y; a_{i_0,0}, \dots, a_{i_{n-1},n-1})\}_{(i_0,\dots,i_{n-1})\in I} \cup \{\neg\varphi(y; a_{j_0,0}, \dots, a_{j_{n-1},n-1})\}_{(j_0,\dots,j_{n-1})\in J}$$

Thus $\mathrm{Th}(K'_\omega)$ has the IP_n property by compactness. As again K'_ω is interpretable in K, we can conclude that the theory of K has the IP_n property as well. $\square$

Corollary 7.4. *The theory of any non separably closed PAC field has the IP_n property.*

In the special case of pseudo-finite fields or, more generally, e-free PAC fields the previous corollary is a consequence of a result of Beyarslan proved in [2], namely that one can interpret the n-hypergraph in any such field.

8 Applications to valued fields

In [16] the authors deduce that an NIP valued field of positive characteristic p has to be p-divisible simply by the fact that infinite NIP fields are Artin-Schreier closed [16, Proposition 5.4]. Thus their result generalizes to our framework.

For the rest of the section, we fix some natural number n.

Corollary 8.1. *If (K, v) is an n-dependent valued field of positive characteristic p, then the value group of K is p-divisible.*

Together with Corollary 6.4, we can conclude the following analogue to [16, Corollary 5.10].

Corollary 8.2. *Every n-dependent valued field of positive characteristic p whose residue field is perfect, is Kaplansky, i.e.*

- *the value group is p-divisible;*
- *the residue field is perfect and does not admit a finite separable extension whose degree is divisible by p.*

Now, we turn to the question whether an n-dependent henselian valued field can carry a nontrivial definable henselian valuation. Note that by a definable henselian valuation v on K we mean that the valuation ring of (K, v), i. e. the set of elements of K with non-negative value, is a definable set in the language of rings. We need the following definition:

Definition 8.3. Let K be a field. We say that its absolute Galois group is *universal* if for every finite group G there exist a finite extensions L of K and a Galois extension M of L such that $\mathrm{Gal}(M/L) \cong G$.

As any finite extension of an n-dependent field K of characteristic $p > 0$ is still n-dependent and of characteristic p, one cannot find a finite extensions $L \subseteq M$ of K such that their Galois group $\mathrm{Gal}(M/L)$ is of order p. Hence any n-dependent field of positive characteristic has a non-universal absolute Galois group. Note that Jahnke and Koenigsmann showed in [15, Theorem 3.15] that a henselian valued field whose absolute value group is non universal and which is neither separably nor real closed admits a non-trivial definable henselian valuation. Hence this gives the following result which is a generalization of [15, Corollary 3.18]:

Proposition 8.4. *Let (K, v) be a non-trivially henselian valued field of positive characteristic p which is not separably closed. If K is n-dependent then K admits a non-trivial definable henselian valuation.*

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